Functional Analysis

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Lecture 8

Orthogonal projection II

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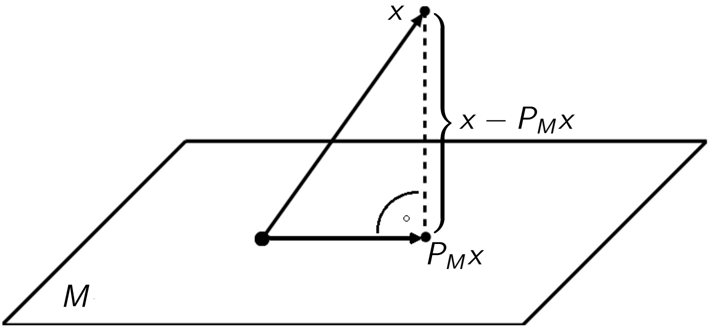
H – fixed Hilbert space. The **orthogonal projection** of $x \in H$ onto subspace $M \subseteq H$ is $P_M x \in M$ such that $x - P_M x \perp M$.



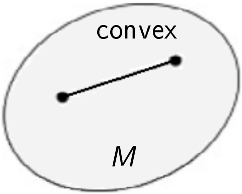
Problem: Does the orthogonal projection $P_M x$ exist?

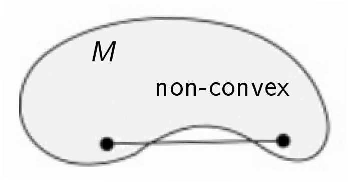
The drawing suggests that

$$\|x - P_M x\| = dist(x, M) := \inf_{y \in M} \|x - y\|.$$

Important: The above infimum is realized at exactly one point,if M is a closed convex set!

A set $M \subseteq H$ is **convex**, if $\forall_{x,y \in M} \forall_{\lambda \in [0,1]} \lambda x + (1-\lambda) y \in M$, that is if every interval with endpoint is M is contained in M:





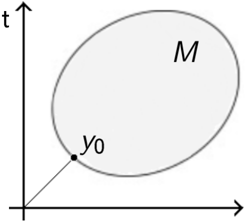
Thm. (On Distances from a Convex Set)

Let H be a Hilbert space. For any closed and convex $M \subseteq H$ and a point $x \in H$ there is a unique $y_0 \in M$ such that $\|x - y_0\| = dist(x, M)$.

Proof: "By moving" the set M we can assume that $x = 0$. Then the theorem takes the form:

In a closed and convex set M there is exactly one element with minimal norm:

$$\exists!_{y_0 \in M} \ \|y_0\| = \inf_{y \in M} \|y\|.$$



"**Existence**". Let $d := \inf_{y \in M} \|y\|$ and let $\{y_n\}_{n=1}^{\infty} \subseteq M$ such that $\|y_n\| \to d$. It suffices to show that $\{y_n\}_{n=1}^{\infty}$ is Cauchy, as then by completeness of M, the sequence is convergent to some $y_0 \in M$ and then $\|y_0\| = \lim_{n\to\infty} \|y_n\| = d$.

$$
\begin{aligned}
\|y_n - y_m\|^2 &\overset{\text{parallelogram}}{\underset{\text{law}}{=\!=\!=}} 2\|y_n\|^2 + 2\|y_m\|^2 - \|y_n + y_m\|^2 \\
&= 2\|y_n\|^2 + 2\|y_m\|^2 - 4\left\|\tfrac{y_n+y_m}{2}\right\|^2 \qquad \begin{cases} \frac{y_n+y_m}{2} \in M \\ \text{as } M \text{ convex} \end{cases} \\
&\leqslant 2\|y_n\|^2 + 2\|y_m\|^2 - 4d^2 \overset{n,m\to\infty}{\longrightarrow} 2d^2 + 2d^2 - 4d^2 = 0.
\end{aligned}
$$

"**Uniqueness**". If $y_1, y_2 \in M$ are such that $\|y_1\| = \|y_2\| = d$, then the above calculations show that $y_1 = y_2$. Indeed

$$
\begin{aligned}
\|y_1 - y_2\|^2 &\overset{\text{parallelogram}}{\underset{\text{law}}{=\!=\!=}} 2\|y_1\|^2 + 2\|y_2\|^2 - \|y_1 + y_2\|^2 \\
&= 2\|y_1\|^2 + 2\|y_2\|^2 - 4\left\|\tfrac{y_1+y_2}{2}\right\|^2 \qquad \begin{cases} \frac{y_n+y_m}{2} \in M \\ \text{as } M \text{ convex} \end{cases} \\
&\leqslant 2\|y_1\|^2 + 2\|y_2\|^2 - 4d^2 = 2d^2 + 2d^2 - 4d^2 = 0.
\end{aligned}
$$

■

Thm. On the existence of orthogonal projection

For a closed subspace $M \subseteq H$ of the Hilbert space H and a point $x \in H$ there exists an orthogonal projection $y = P_M x$. Moreover

$$\|x - y\| = dist(x, M) \tag{1}$$

and the vector y is determined uniquely by this equality.

Proof: Since M is convex, by the previuous theorem there is exactly one $y \in M$ satisfying (1).

We need to show that $x - y \perp M$. Let $z \in M$. For $t \in \mathbb{F}$ we have

$$\|x - y\|^2 \underset{y+tz \in M}{\overset{(1)}{\leqslant}} \|x - (y + tz)\|^2 = \|(x - y) + tz\|^2$$
$$= \|x - y\|^2 - 2\,\mathrm{Re}\,\langle x - y, tz\rangle + |t|^2\|z\|^2.$$

Hence $0 \leqslant |t|^2\|z\|^2 - 2\,\mathrm{Re}\,\langle x - y, tz\rangle$ for $t \in \mathbb{F}$. Putting $t = se^{i\varphi}$, where $s \in \mathbb{R}$ and $\varphi := \arg\langle x - y, z\rangle$ this inequality assumes the form

$$0 \leqslant s^2|e^{i\varphi}|^2\|z\|^2 - 2s\,\mathrm{Re}(e^{-i\varphi}\langle x - y, z\rangle) = s^2\|z\|^2 - 2s|\langle x - y, z\rangle|.$$

So the quadratic function $f(s) = s^2\|z\|^2 - 2s|\langle x-y, z\rangle|$ is nonnegative. Since $f(0) = 0$, its discriminant $\Delta = 4|\langle x-y, z\rangle|^2$ has to be zero. Thus $\langle x-y, z\rangle = 0$, that is $x - y \perp z$. ■

Cor. (Hilbert space decomposition) For any closed subspace M of the Hilbert space H we have

$$H = M \oplus M^{\perp},$$

that is $\forall_{x\in H}\ \exists!_{y\in M}\exists!_{z\in M^{\perp}}\ x = y + z$.

Proof: Let $x \in H$. Put $y := P_M x$ and $z := x - y$. Then $x = y + z$ and from the definition of projection we have $y \in M$ and $x - y \perp M$, i.e. $z \in M^{\perp}$. To show the uniqueness of this decomposition let us assume that $x = y' + z'$ for some $y' \in M$ and $z' \in M^{\perp}$. Then

$$y - y' = z' - z.$$

But $y - y' \in M$, $z - z' \in M^{\perp}$ and $M \cap M^{\perp} = \{0\}$ (zero is the only isotropic vector). Hence $y = y'$ and $z = z'$. ■

Rem. The above corollary can also be written as

$$1 = P_M + P_{M^\perp},$$

where 1 is the identity operator on H and P_M is the map $H \ni x \to P_M x \in M \subseteq H$. In particular, if P_M is the orthogonal projection onto a closed subspace M, then $1 - P_M$ is the projection onto its orthogonal complement $M^\perp$:

$$P_{M^\perp} = 1 - P_M.$$

Cor. $(M^\perp)^\perp = M$ for any closed subspace $M \subseteq H$.

Proof: $P_{(M^\perp)^\perp} = 1 - P_{M^\perp} = 1 - (1 - P_M) = P_M$. ■

Prop. The orthogonal projection P_M is a bounded linear operator of norm 1 (unless $M = \{0\}$ and then $P_M \equiv 0$).

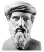
Proof: ...

"**Linearity**". Let $x, y \in H$ and $\alpha, \beta \in \mathbb{F}$. We want to show that $P_M(\alpha x + \beta y) = \alpha P_M x + \beta P_M y$. By definition $P_M(\alpha x + \beta y)$ is the unique element in M such that $(\alpha x + \beta y) - P_M(\alpha x + \beta y) \perp M$. It is therefore sufficient to show that the vector $\alpha P_M x + \beta P_M y$ has the same properties. It is clear that $\alpha P_M x + \beta P_M y \in M$ as M is a linear space. From the linearity of the inner product, for $z \in M$ we get

$$\begin{aligned}\langle(\alpha x + \beta y) - (\alpha P_M x + \beta P_M y), z\rangle &= \alpha\langle x - P_M x, z\rangle + \beta\langle y - P_M y, z\rangle \\ &= 0,\end{aligned}$$

because $x - P_M x$ and $y - P_M y$ are orthogonal to M by definition of projection. Hence $(\alpha x + \beta y) - (\alpha P_M x + \beta P_M y) \perp M$.

"**Boundedness**". For any $x \in H$ we have



$$\|P_M x\|^2 \leqslant \|P_M x\|^2 + \|P_{M^\perp} x\|^2 \underset{Pitagoras}{=} \|P_M x + P_{M^\perp} x\|^2$$

$$\overset{1 = P_M + P_{M^\perp}}{=} \|x\|^2.$$

Thus $\|P_M\| \leqslant 1$. If $P_M \neq 0$, then $M \neq \{0\}$ and there is $x \in M$ with norm 1. Since $P_M x = x$, we get $\|P_M x\| = \|x\| = 1$, and so $\|P_M\| \geqslant 1$. Hence $\|P_M\| = 1$. ■

Thm. Let $P : H \to H$ be a linear idempotent, that is $P^2 = P$. TCAE:

1. P is an orthogonal projection (onto PH),
2. P is self-adjoint, i.e. $\forall_{x,y\in H}\ \langle Px, y\rangle = \langle x, Py\rangle$ ($P = P^*$),
3. P is a contraction, i.e. $\|P\| \leqslant 1$ (more precisely $\|P\| = 1$ or $P = 0$).

Proof: (1)⇒(2). For $x, y \in H$ we have

$$\langle Px, y\rangle \overset{y=Py+(1-P)y}{=} \langle Px, Py\rangle + \langle Px, (1-P)y\rangle \overset{PH\perp(1-P)H}{=} \langle Px, Py\rangle$$
$$\overset{PH\perp(1-P)H}{=} \langle Px, Py\rangle + \langle (1-P)x, Py\rangle \overset{x=Px+(1-P)x}{=} \langle x, Py\rangle.$$

(2)⇒(3). For any $x \in H$ we get

$$\|Px\|^2 = \langle Px, Px\rangle \overset{(2)}{=} \langle P(Px), x\rangle \overset{P^2=P}{=} \langle Px, x\rangle \overset{Schwartz}{\leqslant} \|Px\| \cdot \|x\|.$$

(3)⇒(1). We need to show that $x - Px \perp PH$ dla $x \in H$. 

Hint: Let $x \in \ker P$ and $y \in PH$. Then for $t \in \mathbb{F}$ we get

$$\|y\|^2 = \|Py + tPx\|^2 = \|P(y+tx)\|^2 \overset{\|P\|\leqslant 1}{\leqslant} \|y+tx\|^2 = \|y\|^2 + 2\,\mathrm{Re}\, t\langle x, y\rangle + |t|^2\|x\|^2.$$

Hence $f(s) = 2s|\langle x, y\rangle| + s^2\|x\|^2 \geqslant 0$, which implies that $\langle x, y\rangle = 0$.

Let $H = L^2(\mu)$, where (Ω, Σ, μ) is a measure space.

Ex. (Multiplication by an indicator function)

If $A \in \Sigma$, then $M := \{f \in L^2(\mu) : f \text{ is zero outside } A\}$ is a closed subspace of H, and the orthogonal projection from $L^2(\mu)$ onto M is the operator of multiplication by the indicator function $\mathbb{1}_A$ of A:

$$P_M f = 1_A \cdot f, \qquad f \in L^2(\mu)$$

Ex. (Conditional expectation)

If $\mathcal{F}$ is a σ-subalgebra of Σ, $M := \{f \in L^2(\mu) : f \text{ is } \mathcal{F}\text{-measurable}\}$ is a closed subspace of H. The orthogonal projection from $L^2(\mu)$ onto M. in probability theory is called the **conditional expected value** with respect to $\mathcal{F}$

$$P_M f = E(f, \mathcal{F}), \qquad f \in L^2(\mu).$$

(1) $P_M f$ is $\mathcal{F}$-measurable for every $f \in L^2(\mu)$,

(2) $\int_A P_M f \, d\mu = \int_A f \, d\mu$ for $A \in \mathcal{F}$ and $f \in L^2(\mu)$.